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ABSTRACT

We show an exact analysis of the main parameters that characterize the properties of the Capetanakis-Tsybakov-Mikhailov collision resolution algorithm with the continuous input protocol. In particular we precisely quantify the distributions of the collision resolution interval, the delay experienced by a packet and the state of the top level of the stack that is maintained by the algorithm.

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We show an exact analysis of the main parameters that characterize the properties of the Capetanakis-Tsybakov-Mikhailov collision resolution algorithm with the continuous input protocol. In particular we precisely quantify the distributions of the collision resolution interval, the delay experienced by a packet and the state of the top level of the stack that is maintained by the algorithm.

RESUME:

Nous caractérisons de manière exacte les principaux paramètres du protocole Capetanakis-Tsybakov-Michailov à arrivées continues. Nous obtenons notamment des évaluations précises de la distribution des intervalles de résolution de collision, du délai subi par un paquet et de l'état d'occupation du canal.

1. INTRODUCTION

This paper analyses the performance of a protocol for managing the use of a ***single-channel packet switching communications network*** like the one used in the ALOHA network. We briefly recall the salient features that underlie the ALOHA channel as they apply to the analysis we present below:

a) A single *error-free channel* (bus, cable, broadcast, ...) is shared among many *users* (sources, nodes, stations, ...) which transmit "packetized" *messages*. Time is *slotted* and may be considered discrete. Users are synchronized with respect to the slots. Each slot has a fixed duration equal to the time required to transmit a packet and transmission of packets starts at the beginning of slots only.

b) Each transmission is within the reception range of every user. When more than one user transmit simultaneously, packets are said to *collide* (interfere), and none is transmitted correctly: these collisions are treated as transmission errors and each user strives to retransmit its colliding packet till it is correctly received. The users all employ the same algorithm for this purpose, and have to resolve this contention without the benefit of any other source of information on other users' activity save the common channel.

The collision resolution algorithm is clearly a major determinant of the behaviour of such a transmission process, affecting i.a. the delay experienced by messages until they are successfully transmitted, the buffering requirements at the nodes that maintain the broadcast activity and the efficiency of this multiple access scheme in terms of the maximum traffic rate it will allow before destabilizing.

Out of the plethora of access protocols and *collision resolution algorithms* (CRA) that have been proposed we consider a specific one defined below.

1.1 The Capetanakis-Tsybakov-Mikhailov (C.T.M.) protocol

The analysis of this paper focuses on the Capetanakis-Tsybakov-Mikhailov (C.T.M.) CRA coupled with free, or continuous access of newly arriving packets into the contention. This scheme proves ergodic as long as the the rate of generation of new packets is below a certain bound. This algorithm was first proposed Tsybakov and VVedenskaya [TV81] (see also [Ma82] for a survey of germane issues); our description follows [FH82], where it is described as protocol number 3, and [FFH82]. We note that this variant does not require external control to stabilize and allows users to transmit a packet as soon as it is generated regardless of the state of the channel. Users are therefore not required to monitor the channel continuously but only when they start to transmit; we call those "active users".

Broadly speaking, this protocol is a "divide and conquer" algorithm: when users collide, they separate (recursively) according to some randomization procedure into two groups. Users of the first group attempt retransmission in the next slot while those of the second group wait until the first group has resolved its collisions before they start transmitting again. The CTM algorithm is conveniently implemented by having each user maintain a virtual *stack*. The specification of its use follows:

> The CTM algorithm applies to a channel for which points a) and b) above hold. In particular each active user monitors the channel and knows at the end of a slot if that slot produced a collision or not.
>
> c) Each active user maintains a conceptual stack; at each slot end it determines its position in the stack according to the following procedure (identical for all users which are otherwise unable to communicate their stack state):
>
> – When an inactive user becomes active, it enters level 0 in the stack. It will transmit in the nearest slot and will always do so when at stack level 0.
>
> – At slot end, if it was not a collision slot, a user in stack level 0 (there can be at most one such user, system-wide) becomes inactive, and all other users decrease their stack level counter by 1.
>
> – At slot end, if it was a collision slot, all users at stack level i, $i \geq 1$ change to level $i+1$. The users at level 0 are split into two groups; one remains at level 0, while the other pushes itself into level 1.

It is this splitting that ensures that all users in level 0 will get their packets through before those at lower levels, and this provides for the stability of the channel. The partition can be made on the basis of a random variable such as the flipping of a coin, on the basis of the time when the user became active etc... . We assume however that each user, independently of the other users has the same probability p of staying at level 0 (or probability $q=1-p$ of having to wait at level 1), and these common probabilities are not varied.

The channel capacity of the CTM protocol has been characterized analytically in [FFH82].For $p=\frac{1}{2}$ λ_{max} has been found to be 0.36017.. . Some of the results below borrow from that analysis.

1.2 Characteristic parameters

We are interested here in several characteristic parameters of the CTM algorithm:

> (i) The time it takes to dispose under this algorithm of a group of n users initially in level 0 is termed the *collision resolution interval* (CRI) duration and is denoted by L_n. This includes the slot of the initial collision and subsequent slots until all other active users return to the positions they occupied in their stacks at the collision slot. Note that L_n is generally longer than the time to the n-th successful transmission because of the possibility of newly arriving packets that will be cleared before the CRI is over and because of the possibility of empty levels in other active user stacks that have to be disposed of by "silent" slots.

(ii) The *delay* experienced by a packet is the time from the beginning of its first transmission to the end of its successful one (the two may coincide) and is denoted by W. $W(n)$ denotes the delay of a packet that had its first transmission in a group of n $(n \geq 1)$. Clearly $W(1)=1$, $W(n)>1$ for $n>1$.

(iii) Let Q denote the number of active users at level 0 in the aggregate stack (the union of stacks of all active users). We define the probability distribution $\{q_k\}$ by:

$$q_k = \mathrm{Prob}(Q=k), \qquad k \geq 0.$$

The process governing the creation of new active users is assumed to be Poisson, consistently with the assumption of a large (infinite) transmitter population. The number of newly created active users in each slot is independent of the state of the stack and its history; it is denoted by X and has the Poisson distribution with parameter λ:

$$\mathrm{Prob}(X=i) \equiv a(i) = e^{-\lambda}\frac{\lambda^i}{i!}.$$

1.3 Methodological Note

In developing the analysis it was found expedite to employ two different approaches in generating equations for the variables of interest. The first focuses on a single CRI starting with a given number of initial colliders, n, and uses the protocol specification to provide recursive relations between random variables. These relations are converted to equations in *probability generating functions (pgf's)* that can ultimately be solved explicitly and therefore characterize the behaviour of variables conditioned on the "order" of the CRI – the size of the collision in its initial slot. This is the way taken in Sections 2 and 5. Sections 3 and 4 in addition rely on the fact that the transmission process evolution, when stable, displays recurrent renewal points (e.g. when there are no active users). This allows us to employ the Law of Large Numbers (LLN) and derive relations concerning the steady state behaviour of the channel. What one does then is to sum *the means of variables* such as delay accrued by packets, or accumulated population of specified parts of the system over a renewal interval (usually a CRI) and divides it by similar other sums. The LLN provides the equality of these ratios to the expected value of the ratios of the corresponding random variables, which in principle could have been obtained from the dynamics of the CRA, but frequently at much more computational labour, as the means satisfy simpler relations.

Outline of the paper: In Section 2 we briefly describe a functional equation which will prove ubiquitous throughout the analysis, and provide its solution. This equation was dealt with *in extenso* in [FFH82]. Section 3 provides a direct derivation of the mean delay $E(W)$ using the renewal type of argument mentioned above as developed by Jacquet [Ja83]. In Section 4 we derive the probability mass function (pmf $\{q_k\}$ which is used to derive the variance of W. In section 5, we compute an equation that generates all moments of $W(n)$. Section 6 concludes with approximations for low input rate, numerical examples and discussion.

2. THE BASIC EQUATION AND CRI DURATION

In this section we derive and solve a functional equation that will recur throughout the paper. It is introduced via an analysis of the CRI duration conditioned by n, the number of initial colliders. In particular, the equation will provide us with the mean and variance of the CRI duration under continuous input.

2.1 The distribution of conditional CRI duration

The specification of the CTM-CRA yields the following relation for L_n:

$$L_n = \begin{cases} 1 & \text{if } n=0,1 \\ 1+L_{I+X_1}+L_{n-I+X_2} & \text{if } n\geq 2. \end{cases} \tag{2.1}$$

where I has the binomial distribution $B_p(n)$, X_1 and X_2 are ~Poisson(λ) and the three are independent. Define the generating functions:

$$\begin{array}{lll} (i) & \alpha_n(z)=\sum_{i\geq 1}\text{Prob}(L_n=i)z^i & (\alpha_0(z)=\alpha_1(z)=z) \\ (ii) & \alpha(z,u)=\sum_{n\geq 0}\alpha_n(z)\frac{u^n}{n!} & (\alpha(1,u)=e^u,\ \alpha(z,0)=z) \\ (iii) & \beta(z,u)=e^{-u}\alpha(z,u) & (\beta(1,u)=1,\ \beta(z,0)=z) \end{array} \tag{2.2}$$

Taking the measure of (2.1), we obtain the following sequence which we present once rather in detail:

$$\alpha_n(z)=\sum_{i\geq 1}z^i\sum_{x,y\geq 0}a(x)a(y)\sum_{j=0}^{n}\binom{n}{j}p^jq^{n-j}\,\text{Prob}(L_{j+x}+L_{n-j-y}=i-1).$$

The convolution produces:

$$\alpha_n(z) = z\sum_{x,y\geq 0}a(x)a(y)\sum_{j=0}^{n}\binom{n}{j}p^jq^{n-j}\alpha_{j+x}(z)\alpha_{n-j+y}(z). \tag{2.3}$$

Then from (2.2*ii*)

$$\begin{aligned}\alpha(z,u)-z(u+1) &= z\sum_{\substack{n\geq 2\\ x,y\geq 0}}\frac{u^n}{n!}a(x)a(y)\sum_{j=0}^{n}\binom{n}{j}p^jq^{n-j}\alpha_{j+x}(z)\alpha_{n-j+y}(z) \\ &= z\sum_{n,x,y\geq 0}\frac{u^n}{n!}a(x)a(y)\sum_{j=0}^{n}\binom{n}{j}p^jq^{n-j}\alpha_{j+x}(z)\alpha_{n-j+y}(z)\end{aligned}$$

$$-z\sum_{x,y\geq 0} a(x)a(y)[\alpha_x(z)\alpha_y(z)+uq\,\alpha_x(z)\alpha_{y+1}(z)+up\,\alpha_{x+1}(z)\alpha_y(z)]$$

$$=ze^{-2\lambda}[\sum_{j,x\geq 0}\frac{(up)^j}{j!}\frac{\lambda^x}{x!}\alpha_{j+x}(z)][\sum_{n,y\geq 0}\frac{(uq)^n}{n!}\frac{\lambda^y}{y!}\alpha_{n+y}(z)]$$
$$-ze^{-2\lambda}\alpha(z,\lambda)[\alpha(z,\lambda)+u\,\alpha_u(z,\lambda)] \tag{2.4}$$

where α_u is the partial derivative with respect to u, and using (2.2*iii*), we obtain:

Lemma 1: The modified generating function β of the probability distributions of L_n satisfies the non-local functional equation:

$$\frac{1}{z}\beta(z,u)=\beta(z,\lambda+pu)\beta(z,\lambda+qu)+e^{-u}(u+1)$$
$$-e^{-u}\beta(z,\lambda)[(1+u)\beta(z,\lambda)+u\,\beta_u(z,\lambda)] \tag{2.5}$$

Remark: An alternative view of the β function is that of a *mixture* of the pgf's α_n's; analytical properties of $\beta(\cdot)$ can then reveal the underlying structure of the pgf's.

2.2 Mean value analysis

Equation 2.5 is the starting point for the evaluation of moments of L_n. Consider first the mean values:

$$l_n=E(L_n);$$

and the modified exponential generating function

$$\varphi(u)=e^{-u}\sum_{n\geq 0} l_n\frac{u^n}{n!} \tag{2.6i}$$

where

$$\varphi(u)=\frac{\partial}{\partial z}\beta(z,u)\;|_{z=1} \tag{2.6ii}$$

Differentiating (2.5) with respect to z:

$$-\frac{1}{z^2}\beta(z,u)+\frac{1}{z}\beta_z(z,u)=\beta_z(z,\lambda+up)\beta(z,\lambda+uq)+\beta(z,\lambda+up)\beta_z(z,\lambda+uq)$$
$$-e^{-u}\left\{\beta_z(z,\lambda)[\beta(z,\lambda)(1+u)+u\,\beta_u(z,\lambda)]+\beta(z,\lambda)[\beta_z(z,\lambda)(1+u)+u\,\beta_{uz}(z,\lambda)]\right\} \tag{2.7}$$

and substituting $z=1$, (using (2.2*iii*) and (2.6)):

$$\varphi(u)-1=\varphi(\lambda+pu)+\varphi(\lambda+qu)-e^{-u}[2\varphi(\lambda)(1+u)+u\,\varphi'(\lambda)]. \tag{2.8}$$

To simplify further analysis, we introduce some notation.

Notation:

$$\sigma_1(u)=\lambda+pu \qquad \sigma_2(u)=\lambda+qu \tag{2.9}$$

$$\mathbf{R}f(u) = f(u)-f(\sigma_1(u))-f(\sigma_2(u)) \tag{2.10}$$

furthermore, wherever it matters we shall assume $p\geq q$. Equation (2.8) may then be rewritten as

$$\mathbf{R}\varphi(u) = 1-e^{-u}[2\varphi(\lambda)(1+u)+u\varphi'(\lambda)] \tag{2.11}$$

Substituting $u=0$ in (2.11) and its derivative yields $\varphi(0)=1$, $\varphi'(0)=0$.

A convenient simplification accrues by noting that at $u=\frac{\lambda}{p}$ one has $\sigma_1(u)=2\lambda$, $\sigma_2(u)=u$, and at $u=\frac{\lambda}{q}$ similarly $\sigma_1(u)=u$, $\sigma_2(u)=2\lambda$. Eliminating $\varphi(2\lambda)$ between the resulting equations obtained by these substitutions in (2.11) produces:

$$\varphi'(\lambda)=2\varphi(\lambda)(K-1), \quad K=-\frac{e^{-\frac{\lambda}{p}}-e^{-\frac{\lambda}{q}}}{\frac{\lambda}{p}e^{-\frac{\lambda}{p}}-\frac{\lambda}{q}e^{-\frac{\lambda}{q}}}, \quad K(p=\tfrac{1}{2})=\frac{1}{1-2\lambda}. \tag{2.12}$$

hence

$$\mathbf{R}\varphi(u)=1-2\varphi(\lambda)(1+Ku)e^{-u}. \tag{2.13}$$

Functional equations of this sort can be solved explicitly using the following lemma from [FFH82]:

Lemma 2: The equation $\mathbf{R}g(z)=f(z)$ with specified values for $g(0)$ and $g'(0)$, where $f(z)$ is an entire function that satisfies the consistency requirement $f(\lambda/p)=f(\lambda/q)$ and $p+q=1$, $p,q>0$ has the unique regular solution

$$g(z) = g(0)+g'(0)z + \sum_{\sigma\in H}[f(\sigma(z))-f(\sigma(0))-(p;q)^{\sigma}zf'(\sigma(0))], \tag{2.14}$$

where H is the (non-commutative) semigroup generated by composition of σ_1 and σ_2, $|\sigma|_i$ is the number of occurrences of σ_i in σ and

$$(p;q)^{\sigma}=p^{|\sigma|_1}\cdot q^{|\sigma|_2}.$$

In the sequel, we denote by $\mathbf{S}(f(\cdot);z)$ the sum appearing in (2.14):

$$\mathbf{S}(f(\cdot);z) = \sum_{\sigma\in H}[f(\sigma(z))-f(\sigma(0))-(p;q)^{\sigma}zf'(\sigma(0))].$$

The above lemma expresses φ as a sum indexed over the semigroup H:

$$\varphi(z)=1-2\varphi(\lambda)\mathbf{S}(e^{-u}(1+Ku);z). \tag{2.15}$$

Simply substituting $u=\lambda$ in the solution,we get

$$\varphi(\lambda)=1-2\varphi(\lambda)\mathbf{S}(e^{-u}(1+Ku);\lambda). \tag{2.16}$$

This last relation is a linear equation in $\varphi(\lambda)$ which can be solved explicitly. From there, one gets an expression for $\varphi(z)$ by (2.14) that no longer involves $\varphi(\lambda)$:

Theorem 1: The modified exponential generating function of the mean values of the conditioned CRI length

$$\varphi(z) = e^{-z} \sum_{n \geq 0} l_n \frac{z^n}{n!}$$

is given by:

$$\varphi(z) = 1 - 2\varphi(\lambda) \mathbf{S}(e^{-u}(1+Ku); z)$$

where

$$\varphi(\lambda) = \frac{1}{1+2\mathbf{S}(e^{-u}(1+Ku);\lambda)} . \tag{2.17}$$

This leads to an explicit form of the Taylor coefficients of φ, $\varphi_j = [u^j]\varphi(u)$, and through them for the l_n since:

$$l_n = n! \sum_{j=0}^{n} \frac{\varphi_j}{(n-j)!} .$$

In this way, one obtains the estimates of the mean (conditioned) CRI length. In [FFH82], the asymptotic behaviour of l_n is also investigated via this expression.

2.3 Variance estimates

Differentiating equation (2.7) once more with respect to z, at $z=1$, we obtain a similar equation for $\beta_{zz}(1,u) \equiv \psi(u)$ which is required later:

$$\begin{aligned} \mathbf{R}\psi(u) &= 2\varphi(u) - 2 + 2\varphi(\sigma_1(u))\varphi(\sigma_2(u)) \\ &- e^{-u}\{ 2(1+u)\psi(\lambda) + u\psi'(\lambda) + 2\varphi^2(\lambda)[1+u(2K-1)]\} \quad \psi(0) = \psi'(0) = 0. \end{aligned} \tag{2.18}$$

The same elimination procedure used for equation (2.11) can be affected here, leading to

$$\psi'(\lambda) = 2\psi(\lambda)(K-1) - 2\varphi^2(\lambda)(K-1) + \overline{K}, \quad \overline{K} = 4\varphi(\lambda) \frac{(1+K\frac{\lambda}{p})[\varphi(\frac{\lambda}{q}) - \varphi(\frac{\lambda}{p})]}{[\frac{\lambda}{q}e^{-\frac{\lambda}{q}} - \frac{\lambda}{p}e^{-\frac{\lambda}{p}}]\, e^{-\lambda/p}} . \tag{2.19}$$

$$\overline{K}(p = 1/2) = 4\varphi(\lambda)\varphi'(2\lambda) / (1-2\lambda)^2$$

hence

$$\mathbf{R}\psi(u) = a(u) - 2\psi(\lambda) e^{-u}(1+Ku) \tag{2.20}$$

where

$$a(u) = 2\varphi(u) - 2 + 2\varphi(\sigma_1(u))\varphi(\sigma_2(u)) - e^{-u}\{2\varphi^2(\lambda)(1+uK) + u\overline{K}\}, \tag{2.21}$$

Equation (2.20) can now be solved precisely in the same way as (2.13) (using the solution of the latter). Hence:

Theorem 2: The modified exponential generating function of the second moments of conditional CRI duration satisfies:

$$e^{-z}\sum_{n\geq 0} E(L_n^2)\frac{z^n}{n!} = \psi(z)+\varphi(z)$$

where

$$\psi(z) = S(a(u);z)+(\varphi(z)-1)S(a(u);\lambda),$$

and $a(u)$ is given by equation (2.21). The theorem follows since one can extract the coefficients $\psi_j = [u^j]\psi(u)$, noting that $\psi(u) = e^{-u}\sum_{n\geq 0}\frac{u^n}{n!}E(L_n^2-L_n)$. This then provides the variance of the conditional CRI durations.

2.4 Moments of the Unconditional CRI Duration

A serendipitous result of our using exponential generating functions (egf) and Poisson distributed arrivals is that usually the egfs when evaluated at λ have a probabilistic meaning. For instance, using (2.6i), we have

$$\varphi(\lambda) = \sum_{n\geq 0} a(n)E(L_n) \tag{2.22}$$

and this quantity represents the *unconditional mean length* of a CRI duration. Proceeding similarly with the variance estimates, we have obtained:

Theorem 3: The mean and variance of unconditional CRI length are:

$$E(L) = \varphi(\lambda) = \frac{1}{1+2S(e^{-u}(1+Ku);\lambda)}$$

$$V(L) = \psi(\lambda) + \varphi(\lambda)(1-\varphi(\lambda))$$

where

$$\psi(\lambda) = \varphi(\lambda)S(a(u);\lambda)$$

and $a(u)$ is defined by equation (2.21).

3. DIRECT EVALUATION OF THE MEAN DELAY

In this Section we show how the mean packet delay can be rather simply evaluated from the recurrent nature of the process of channel congestion and collision resolution. The following is adapted from [Ja83].

Define C_n as the total sojourn time experienced by all users that became active during a CRI which started out as an n-fold collision. (We also consider "degenerate" 1-

collision, and clearly for those $C_1=1$.) Let $E(C_n)\equiv c_n$. Similarly, let b_n be the mean number of such active users.

Defining the egfs

$$c(z) = e^{-z}\sum_{n\geq 1} c_n \frac{z^n}{n!} \tag{3.1}$$

and

$$b(z) = e^{-z}\sum_{n\geq 1} b_n \frac{z^n}{n!} \tag{3.2}$$

we obtain

Theorem 4 The unconditional mean delay of a packet is

$$E(W) = \frac{c(\lambda)}{b(\lambda)} = 1 + \frac{1}{\lambda}\mathbf{S}[qz\varphi(\sigma_1(z)) - zA(\lambda)e^{-z};\lambda] \tag{3.3}$$

with

$$A(\lambda) = \frac{\frac{1}{q}-\frac{1}{p}+\varphi(\frac{\lambda}{q})-\frac{q}{p}\varphi(2\lambda)}{\frac{1}{q}e^{-\frac{\lambda}{q}}-\frac{1}{p}e^{-\frac{\lambda}{p}}}\ , \quad A(\lambda)(p=\tfrac{1}{2}) = e^{2\lambda}\frac{2+\varphi(2\lambda)+\lambda\varphi'(2\lambda)}{2(1-2\lambda)}. \tag{3.4}$$

Proof Firstly we observe that $b(\lambda)=\lambda\varphi(\lambda)$, as an immediate result of the renewal properties of the protocol and (2.22). We proceed to evaluate $c(\lambda)$.

Precisely the same reasoning that produced (2.1) yields for the $\{C_n\}$:

$$C_n = \begin{cases} 0 & n=0 \\ 1 & n=1 \\ n + C_{I+X_1} + C_{n-I+X_2} + (n-I)L_{I+X_1} & n\geq 2 \end{cases} \tag{3.5}$$

Removing the conditioning and summing over n, we get at some labour

$$\mathbf{R}c(z) = z + qz\varphi(\sigma_1(z)) - e^{-z}[qz\varphi(\lambda)+2c(\lambda)(1+z)+zc'(\lambda)] \tag{3.6}$$

The by-now familiar procedure to eliminate the derivative yields

$$c'(\lambda) = 2c(\lambda)(K-1) - q\varphi(\lambda) + A(\lambda), \tag{3.7}$$

with $A(\lambda)$ given in (3.4); thus

$$\mathbf{R}c(z) = z + qz\varphi(\sigma_1(z)) - 2c(\lambda)(1+Kz)e^{-z} - zA(\lambda)e^{-z} \tag{3.8}$$

and using the $\mathbf{S}$ operator and the Lemma of Section 2

$$c(z) = z + \frac{c(\lambda)}{\varphi(\lambda)}(\varphi(z)-1) + \mathbf{S}[qu\varphi(\sigma_1(u)) - uA(\lambda)e^{-u};z] \tag{3.9}$$

The required value $c(\lambda)$ is obtained by replacing z by λ in (3.9) and solving for $c(\lambda)$. The computation is similar to that done for (2.16), but heavier, due to the presence of the φ function in **S**.

Finally, equation (3.3) holds by virtue of the renewal property of the protocol and the identification of $c(\lambda)$ and $b(\lambda)$ as the mean unconditional accumulated delay and number of packets, respectively, incurred during a CRI. ∎

The above procedure yields $E(W)$, but it cannot lead to the values of higher moments, due to the linearity it utilizes. Whereas the variance of the delay is of no smaller interest we have to resort to a more involved analysis, beginning with an examination of the top level in the CRA stack.

4. DISTRIBUTION OF THE STATES OF TOP-OF-THE-STACK

4.1 Preliminaries

Let Q be the multiplicity of a transmission, which we identify with the number of users at level 0 of their respective CRA stacks. Possession of the distribution of Q permits us to condition on it, and since this is where most of the CRA activity occurs one would expect it to permit deeper penetration into the CRA mechanisms than we could hitherto. In particular, we shall see it yields the *distribution* of the delay experienced by individual packets.

Note that Q does not define a Markov chain; evaluating it through the CRA dynamics calls for the consideration of the *entire* stack and its evolution, which leads to quite involved computations.

Instead, we compute the distribution of Q by making use of the Law of Large Numbers (LLN) in the following way: Consider a sequence of N CRIs.[1] The expected number of n−order CRIs among them is $Na(n)$, and their total length has the expected value

$$N\sum_{n\geq 0} a(n)L_n = N\varphi(\lambda) \tag{4.1}$$

Now define

η_n^k − mean number of slots with top-of-stack population of k packets ($\equiv k$−slots) during L_n.

Hence $N\sum_{n\geq 0} a(n)\eta_n^k$ is the expected number of k−slots in the sequence specified above, and by the LLN

[1] Since we view a slot in which there is no activity as a degenerate CRI of order 0 we may say that the time axis is entirely covered by CRIs.

$$\text{Prob}(Q=k)\equiv q_k = \frac{N\sum_{n\ge 0} a(n)\eta_n^k}{N\varphi(\lambda)} \tag{4.2}$$

We remark that the same approach precisely can be used to obtain the distribution of the states of *any* level of the stack, by counting the number of slots during which that level enjoyed different occupancies.

4.2 Top-of-Stack Occupancy Probabilities

Returning to the top-of-stack and evaluating these probabilities we obtain

Theorem 5: The pgf of $\{q_k\}_{k\ge 0}$ is given by

$$q(u) = 1+\lambda(u-1) + \mathbb{S}(t(z,u);z=\lambda) \tag{4.3}$$

where

$$t(z,u) = e^{(u-1)z} - ze^{-z}\frac{e^{\frac{\lambda}{q}(u-1)} - e^{\frac{\lambda}{p}(u-1)}}{\frac{\lambda}{q}e^{-\frac{\lambda}{q}} - \frac{\lambda}{p}e^{-\frac{\lambda}{p}}} \tag{4.4}$$

Furthermore, we can obtain explicitly

$$\begin{aligned} q_0 &= \tfrac{1}{2}-\lambda+\frac{1}{2\varphi(\lambda)} \\ q_1 &= \lambda \\ q_k &= \frac{1}{k!}\mathbb{S}(e^{-z}(z^k - z\,\mathrm{K}_k);\lambda) \qquad k\ge 2 \end{aligned} \tag{4.5}$$

with

$$\mathrm{K}_k = \frac{\left(\frac{\lambda}{q}\right)^k e^{-\frac{\lambda}{q}} - \left(\frac{\lambda}{p}\right)^k e^{-\frac{\lambda}{p}}}{\frac{\lambda}{q}e^{-\frac{\lambda}{q}} - \frac{\lambda}{p}e^{-\frac{\lambda}{p}}}\,, \quad \mathrm{K}_k\,(p=\tfrac{1}{2}) = \frac{(2\lambda)^{k-1}(k-2\lambda)}{1-2\lambda}.$$

Proof: The dynamics of the CRA provide, as before, a linear relation

$$\eta_n^k = \delta_{k,n} + \sum_{j=0}^{n}\binom{n}{j}p^j q^{n-j}\sum_{x,y\ge 0} a(x)a(y)(\eta_{j+x}^k + \eta_{n-j+y}^k) \tag{4.6}$$

which, with the obvious definition $P_n(u) = \sum_{k\ge 0}\eta_n^k u^k$, becomes

$$P_n(u) = \begin{cases} 1 & n=0 \\ u & n=1 \\ u^n + \sum_{j=0}^{n}\binom{n}{j}p^j q^{n-j}\sum_{x\ge 0} a(x)[P_{j+x}(u) + P_{n-j+x}(u)] & n\ge 2 \end{cases} \tag{4.7}$$

Note that in contradistinction with earlier derivations we cannot generate a relation over the sought distribution $\{q_k\}$ itself. From (4.7) however we obtain, having defined

$$P(z,u) = e^{-z}\sum_{n\geq 0} P_n(u)\frac{z^n}{n!}, \quad P(0.u)=1, \quad P_z(0,u)=u-1 \tag{4.8}$$

a familiar equation:

$$\mathbf{R}P(z,u) = ^2 e^{(u-1)z} - e^{-z}[2P(\lambda,u)(1+z) + zP_z(\lambda,u)] \tag{4.9}$$

Invoking the LLN again, we have that the probabilities $\{q_k\}$ have the pgf

$$q(u) = \frac{P(\lambda,u)}{\varphi(\lambda)} \tag{4.10}$$

To compute this function first note that equation (4.9) can be reduced in the accepted manner to

$$\mathbf{R}P(z,u) = t(z,u) - e^{-z}2P(\lambda,u)(1+Kz) \tag{4.11}$$

with $t(z,u)$ given in (4.4), and using Lemma 2

$$P(\lambda,u) = \varphi(\lambda)[1+\lambda(u-1) + \mathbf{S}(t(z,u);z=\lambda)]$$

whence equation (4.10) immediately provides (4.3). The individual probabilities q_k can be extracted as usual from the pgf. Note that differentiation with respect to u commutes with the operator $\mathbf{S}$. In particular, the values for $k=0$ and 1 are obtained by observing that $t(z,0) = e^{-z}(1+Kz)$ and $\frac{\partial}{\partial u}t(z,u)|_{u=0} = 0$. For $k=0$ we use (2.16). ∎

Note the non-surprising value of q_1.

4.3 Top-of-Stack Observed States.

The random variable Q was defined as the number of packets in a randomly chosen slot. For the evaluation of random variables that depend on this quantity we also need a related random variable, denoted by R, which is the occupancy of this level in which *a random packet is transmitted for the first time*. We obtain

Theorem 6 The probability mass function of R is given by

$$r_k \equiv \text{Prob}(R=k) = q_{k-1} \tag{4.12}$$

Proof Q consists of two components: newly arriving packets (X) and such that either popped from level 1 or remained at level 0 following a partition (Q_0). Since the two are independent we have for the pgf of Q_0, denoted by $q_0(u)$ the value

$$q_0(u) = q(u)e^{\lambda(1-u)} \tag{4.13}$$

A random packet arrives in a group with size distributed as the random variable $\overline{X}$, where $\text{Prob}(\overline{X}=j) = ja(j)/\lambda$, which combine to give the pgf $g_{\overline{X}}(u) = ue^{\lambda(u-1)}$. Since $R = Q_0+\overline{X}$, and the two are independent we have the pgf

---

[2]The functions σ_i operate on z only!

$$g_R = g_0(u) g_X(u) = u q(u) \tag{4.14}$$

and (4.12) follows immediately. ∎

We are now armed enough to attack the main target of this analysis: the distribution of delay of a random packet.

5. MOMENTS OF PACKET DELAY TIME

5.1 Preliminaries

The results of Section 4 will now be utilized to evaluate the variance of the delay experienced by a random packet. En passant we shall also derive anew an expression for the mean of the delay. $W(n)$ is the delay experienced by a packet that had its first transmission in an n–slot. Clearly $W(1)=1$ and $W(n)>1$ for $n>1$. In subsection 5.4 we shall show how an elaboration of the argument used in Sections 3 and 4 could be used to evaluate the second moment directly, at comparable computational effort to the procedure presented below.

5.2 The Distribution of $W(n)$

The dynamics of the CRA provide the relation

$$W(n) = \begin{cases} W(B_p(n-1)+1+X_1) & \text{with probability } p \\ L_{B_p(n-1)+X_2} + W(n-B_p(n-1)+X_3) & \text{with probability } q \end{cases} \tag{5.1}$$

where $B_p(n-1)$ is a random variable with the corresponding binomial distribution, the X_i, $i=1,2,3$ are all Poisson(λ)–distributed, n has the pmf $\{r_n\}$ given in (4.12) and they are all independent. Introduce the notation

$$w(z;n) = \sum_{i\geq 1} P(W(n)=i) z^i; \qquad h(z,u) = e^{-u} \sum_{n\geq 1} w(z;n) \frac{u^{n-1}}{(n-1)!} \tag{5.2}$$

and then (5.1) can be shown to lead to

$$\frac{1}{z} h(z,u) = p h(z,\sigma_1(u)) + q h(z,\sigma_2(u)) \beta(z,\sigma_1(u)) + e^{-u}[1-h(z,\lambda)(p+q\beta(z,\lambda))] \tag{5.3}$$

where $\beta(z,u)$ is as defined in (2.2*iii*). Note that (5.3) does not have the form of our basic equation!

Unfortunately we do not know how to mix the the $w(z;n)$ with the pmf $\{r_n\}$ – else we could have produced a direct relation for the distribution of the unconditional delay. Since anyway the first two moments supply most of the useable information we shall concentrate on these. Another point worth making is that when the transmission process is stable, n–slots with low n predominate: From Theorem 5 we see that $q_0+q_1 = \frac{1}{2}+\frac{1}{2\varphi(\lambda)}$, and taking a particular example, for $\lambda=0.25$ and $p=0.6$ we find $q_0+q_1+q_2 \approx 0.965$. Thus the lack of a proper relation for the unconditional delay causes no hardship from the computational point of view, as the first few $W(n)$ suffice.

5.3 Equations for the first two moments.

To obtain results for $E(W(n))$ we define

$$w_1(u) \equiv \frac{\partial}{\partial z} h(z,u)\,|_{z=1} = e^{-u} \sum_{n\geq 1} E[W(n)]\,\frac{u^{n-1}}{(n-1)!} \tag{5.4}$$

and equation (5.3) provides

$$w_1(u) - pw_1(\sigma_1(u)) - qw_1(\sigma_2(u)) = 1 + q\,\varphi(\sigma_1(u)) - e^{-u}[w_1(\lambda) + q\,\varphi(\lambda)]. \tag{5.5}$$

Similarly defining

$$w_2(u)) \equiv \frac{\partial^2}{\partial z^2} h(z,u)\,|_{z=1} = e^{-u} \sum_{n\geq 1} E[W^2(n)]\,\frac{u^{n-1}}{(n-1)!} - w_1(u) \tag{5.6}$$

one finds that $w_2(\cdot)$ satisfies an equation similar to (5.5):

$$\begin{aligned} w_2(u) - pw_2(\sigma_1(u)) - qw_2(\sigma_2(u)) &= 2qw_1(\sigma_2(u))\varphi(\sigma_1(u)) + q\,\psi(\sigma_1(u)) \\ &\quad + 2w_1(u) - 2 - e^{-u}[w_2(\lambda) + 2qw_1(\lambda)\varphi(\lambda) + q\,\psi(\lambda)]. \end{aligned} \tag{5.7}$$

Since equations (5.5) and (5.7) are not in a form suitable to be solved via Lemma 2, we present a sister Lemma to it:

Lemma 3 The equation $g(z) - pg(\sigma_1(z)) - qg(\sigma_2(z)) = f(z)$, with specified value for $g(0)$ and $f(\cdot)$ entire, has the unique regular solution

$$g(z) = g(0) + \sum_{\sigma\in H} (p;q)^{\sigma}[f(\sigma(z)) - f(\sigma(0))], \tag{5.8}$$

and corresponding to the **S** operator we shall define a **T** operator on entire functions:

$$\mathbf{T}(f(\cdot);z) \equiv \sum_{\sigma\in H} (p;q)^{\sigma}[f(\sigma(z)) - f(\sigma(0))].$$

Lemma 3 provides for a solution of (5.5) and (5.7); the values of $w_i(\lambda)$ are obtained by substitution, precisely as $\varphi(u)$ and $\varphi(\lambda)$ were obtained. Since (5.4) and (5.6) provide

$$E[W(n)] = (n-1)![u^{n-1}]e^{u}w_1(u) \tag{5.9}$$

and

$$E[W^2(n)] = (n-1)![u^{n-1}]e^{u}(w_1(u) + w_2(u)) \tag{5.10}$$

the required moments are immediately available:

$$E[W^k] = \sum_{n\geq 1} r_n E[W^k(n)]. \tag{5.11}$$

Remark: the evaluation of the **T** operator on the right-hand sides of (5.5) and (5.7) calls for heavier computations than so far.

5.4 An approach to W via the LLN

In this subsection we outline an alternative approach to the calculation of the distribution and moments of the unconditional delay. The idea is to define a ***random generating function*** $\mathbf{W}_n(z)$ for a CRI of order n, with $[z^j]\mathbf{W}_n(z)$ being the (random) number of packets that experience a delay of j during a CRI of order n. Letting $W_n(z)$ be the expected value of $\mathbf{W}_n(z)$ we can write a relation for it:

$$W_n(z) = E_{I,X_1,X_2}[W_{I+X_1}(z) + W_{n-I+X_2} + nw(z;n) - Iw(z;I+X_1) - (n-I)w(z;n-I+X_2)] \quad (5.12)$$

Defining as usual $W(z,u) = \sum_{n\ge 1} W_n(z)\frac{u^{n-1}}{(n-1)!}$, the first two derivatives of $W(z,u)$ with respect to z at $z=1$, evaluated at $u=\lambda$ provide the desired moments. Since the calculations are similar to those required in subsection 5.3 we do not proceed with them.

In the next Section certain approximations and numerical results are presented.

6. NUMERICAL RESULTS AND DISCUSSION

6.1 Approximations for small input rate

In the preceding sections, we have obtained explicit expressions for the various parameters of the tree protocol: CRI duration, unconditional waiting time... . The expressions obtained involve combinations of the summation operators $\mathbf{S}$ and $\mathbf{T}$ applied to standard entire functions. A convenient way to evaluate them numerically is to have them expanded as power series in λ, though one may expect reasonably short expansions to yield useful information for moderate values of λ only. To expand expressions such as $\mathbf{S}(f(u);\lambda)$, $\mathbf{T}(f(u);\lambda)$ for entire $f(u)$, it is required to determine the quantities:

$$\mathbf{S}(u^k;\lambda) = \sum_{\sigma\in H}\{\sigma(\lambda)^k - \sigma(0)^k - k(p;q)^\sigma\sigma(0)^{k-1}\} \quad (6.1)$$

$$\mathbf{T}(u^k;\lambda) = \sum_{\sigma\in H}(p;q)^\sigma[\sigma(\lambda)^k - \sigma(0)^k]\,. \quad (6.2)$$

To evaluate these sums, express $\sigma(\lambda)$ as $\sigma(0)+\lambda(p;q)^\sigma$, and the task is now reduced to evaluating sums of the form:

$$\xi_{j,l} = \sum_{\sigma\in H}\sigma(0)^j(p^l;q^l)^\sigma \quad (6.3)$$

since by the binomial theorem we have for $k\ge 2$:

$$\mathbf{S}(u^k;\lambda) = \sum_{l=2}^{k} \binom{k}{l} \xi_{k-l,l}\, \lambda^l \; . \quad \mathbf{T}(u^k;\lambda) = \sum_{l=1}^{k} \binom{k}{l} \xi_{k-l,l+1}\, \lambda^l \; . \tag{6.4}$$

Note, in considering the order of the expansions, that $\sigma(0) = O(\lambda)$. The quantities $\xi_{j,l}$ can be expressed in closed form (for fixed j and l). Using the standard decomposition

$$H = \varepsilon \cup \sigma_1 H \cup \sigma_2 H \, ,$$

we find for $j \geq 1$

$$\xi_{j,l} = p^l \sum_{\sigma \in H} (\lambda + p\,\sigma(0))^j (p^l;q^l) + q^l \sum_{\sigma \in H} (\lambda + q\,\sigma(0))^j (p^l;q^l)$$

whence expanding and solving for $\xi_{j,l}$, when $j \geq 1$:

$$\xi_{j,l} = \frac{1}{1-p^{l+j}-q^{l+j}} \sum_{r=0}^{j-1} \binom{j}{r} \lambda^{j-r} (p^{l+r}+q^{l+r}) \xi_{r,l} \tag{6.5}$$

and

$$\xi_{0,l} = \frac{1}{1-p^l-q^l} . \tag{6.6}$$

In this manner, we can compute the values of the **S** and **T** operators applied to any function with known Taylor expansion around the origin, to any power of λ, by means of equations (4)-(6).

We have programmed the corresponding algorithms in the MACSYMA language for symbolic calculations, and determined expansions for the quantities previously studied. The program is added as an appendix. Thus we find for the expected values of the unconditional CRI duration and packet waiting time:

$$\varphi(\lambda) = 1 + \frac{\lambda^2}{2pq} + \frac{3+8pq}{18p^2q^2}\lambda^3 + O(\lambda^4) \tag{6.7}$$

$$E(W) = 1 + \frac{2-p}{2pq}\lambda + \frac{3+7p-11p^2+4p^3}{6p^2q^2}\lambda^2 + O(\lambda^3). \tag{6.8}$$

Similarly, retaining second order quantities only we find

$$q(u) = 1 - \lambda + \frac{3\lambda^2}{2pq} + u\left(\lambda - \frac{2\lambda^2}{pq}\right) + u^2 \frac{\lambda^2}{2pq} . \tag{6.9}$$

6.2 Numerical examples

We also evaluated the **S** and **T** directly for the various right- hand-sides that occurred in the preceding sections, a procedure that is not limited in its applicability to low values of λ. This produced the enclosed two tables. In the first one we bring channel-oriented quantities, specifically the mean and variance of the unconditional CRI durations. These are the upper and lower entries in each box, respectively. Note that the computed values also account for the contribution of the "degenerate" CRI's, of orders 0 and 1. The missing entries, in both tables, refer to combinations of p and λ which overload the system. Note the symmetry with respect to the value $p = \frac{1}{2}$.

λ ..p	.250	.350	.400	.480	.500	.520	.560	.580	.600	.750
.010	1.00027 .0025	1.00022 .0018	1.00021 .0014	1.00020 .0013	1.00020 .0013	1.00020 .0013	1.00021 .0013	1.00021 .0014	1.00021 .0014	1.00027 .0025
.050	1.00772 .0880	1.00625 .0543	1.00589 .0475	1.00565 .0430	1.00564 .0428	1.00565 .0430	1.00573 .0444	1.00580 .0457	1.00589 .0475	1.00772 .0880
.100	1.03734 .6064	1.02934 .3468	1.02750 .2981	1.02623 .2669	1.02618 .2657	1.02623 .2669	1.02664 .2768	1.02701 .2859	1.02750 .2981	1.03734 .6064
.150	1.10812 2.8284	1.08106 1.4218	1.07516 1.1871	1.07118 1.0422	1.07102 1.0368	1.07118 1.0422	1.07246 1.0879	1.07362 1.1297	1.07516 1.1871	1.10812 2.8284
.200	1.27690 14.5630	1.19010 5.6940	1.17304 4.5124	1.16187 3.8266	1.16143 3.8014	1.16187 3.8266	1.16545 4.0389	1.16868 4.2367	1.17304 4.5124	1.27690 14.5630
.250	1.81301 145.6909	1.44860 29.9839	1.39301 21.2525	1.35876 16.7877	1.35746 16.6319	1.35876 16.7877	1.36954 18.1217	1.37945 19.4046	1.39301 21.2525	1.81301 145.6909
.300	17.98337 615140.00	2.40096 429.3438	2.08520 211.0952	1.92443 135.8407	1.91877 133.5746	1.92443 135.8407	1.97265 156.1545	2.01887 177.4061	2.08520 211.0952	17.98337 615140.0
.350	—— ——	—— ——	44.34207 7.076e+6	8.42348 37108.0	8.17555 33571.0	8.42348 37108.0	11.27045 96904.0	16.41953 323880.0	44.34207 7.076e+6	—— ——

Table 1: Channel characteristics: the mean and variance of CRI durations.

In the second table user-oriented quantities are presented - the probability of a packet to be involved in a collision in its initial transmission, and the mean and variance of the time until its successful transmission.

λ ..p	.250	.350	.400	.480	.500	.520	.560	.580	.600	.750
.010	.010137 1.0486 .413	.010112 1.0376 .236	.010106 1.0346 .195	.010102 1.0316 .168	.010102 1.0311 .153	.010102 1.0308 .148	.010104 1.0304 .143	.010105 1.0303 .142	.010106 1.0303 .142	.010137 1.0351 .195
.050	.053832 1.2934 3.119	.053103 1.2225 1.754	.052929 1.2037 1.457	.052808 1.1862 1.209	.052804 1.1837 1.178	.052808 1.1819 1.156	.052848 1.1803 1.145	.052883 1.1805 1.152	.052929 1.1814 1.170	.053832 1.2225 1.976
.100	.117996 1.7819 12.924	.114251 1.5712 6.816	.113380 1.5190 5.654	.112779 1.4738 4.805	.112756 1.4682 4.732	.112779 1.4648 4.705	.112975 1.4637 4.829	.113149 1.4664 4.944	.113380 1.4714 5.135	.117996 1.6337 11.639
.150	.198787 2.7008 53.431	.187490 2.1681 24.431	.184953 2.0488 19.841	.183224 1.9534 16.952	.183156 1.9436 16.832	.183224 1.9388 16.914	.183784 1.9447 17.916	.184285 1.9558 18.665	.184953 1.9731 19.808	.198787 2.4837 61.511
.200	.308427 4.8020 89.686	.279866 3.3326 100.083	.273758 3.0492 77.180	.269658 2.8446 64.554	.269498 2.8282 64.385	.269658 2.8244 65.212	.270980 2.8560 71.063	.272168 2.8930 75.398	.273758 2.9469 81.901	.308427 4.5993 382.352
.250	.474215 11.9516 3159.805	.404840 6.1641 571.199	.391065 5.3369 393.659	.382016 4.8125 308.747	.381665 4.7831 308.087	.382016 4.7885 314.057	.384913 4.9174 353.371	.387535 5.0463 384.505	.391065 5.2298 431.772	.474215 12.3538 4371.260
.300	.772197 258.0161 1.0179e+7	.591750 17.5785 8323.103	.560215 13.1456 4065.251	.540183 10.9350 2632.735	.539416 10.8550 2611.280	.540183 10.9288 2684.178	.546533 11.5939 3211.629	.552337 12.2443 3700.430	.560215 13.1950 4496.456	.772197 295.7230 1.2319e+7
.350	—— —— ——	—— —— ——	.836724 545.2620 9.2693e+7	.790642 89.5118 5.5794e+5	.788842 86.5718 5.0984e+5	.790642 90.0089 5.6518e+5	.805636 127.6223 1.4519e+6	.819548 195.6292 4.6952e+6	.836724 565.4104 9.6946e+7	—— —— ——

Table 2: User oriented characteristics: Collision probability, $E[W]$, $V(W)$.

The items appear in this order in each box, from top to bottom. Note that while the collision probability (equal to $1-r_1$, of Section 4) is symmetrical with respect to $p=\frac{1}{2}$, this is no longer true for the delay statistics. For low values of λ we find that $E[W]$ is minimized at $p^*=2-\sqrt{2}\approx 0.586$, as can be simply shown by taking the development of $E[W]$ in λ to the first order. Little reflection will show that this can be expected, as $p>\frac{1}{2}$ rather decreases the probability of wasting the *first* slots. As λ increases the importance of efficient splitting overtakes, to reduce the value of p that minimizes the mean delay to very nearly one half, when $\lambda \to \lambda_{max}$. Note the extreme sensitivity of $V(W)$ to the load on the channel.

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APPENDIX: MACSYMA COMPUTATIONS

Commands	Functions
`q:1-p;`	
`(ratvars(ll,zz),ratweight(ll,1,zz,1),ratwtlvl:kmax);`	*set_up*
`xi[j,l]:=if j=0 then 1/(1-p^l-q^l) else 1/(1-p^(l+j)-q^(l+j))*` `sum(binomial(j,r)*rat(ll)^(j-r)*(p^(l+r)+q^(l+r)),r,0,j-1);`	$\xi_{j,l}$
`suz[k]:=sum(binomial(k,l)*xi[k-l,l]*rat(z)^l,l,2,k);`	$S(u^k;z)$

`ssz(f):=block([s,ff],ff:rat(expand(taylor(f,z,0,kmax))),`
`ff:ff- sub st(0,z,ff)-z*coeff(ff,z,1),s:0,`
`for i:2 thru kmax do (s:s+suz[i]*coeff(ff,z,i)),rat(s));`
	$S(f(z);z)$

`ppz[k]:=ssz(exp(-z)*z^k);`	$S(e^{-z}z^k;z)$
`ppl[k]:=rat( sub st(ll,z,ppz[k]));`	$S(e^{-z}z^k;\lambda)$
`expkmax:taylor(exp(x),x,0,kmax+1);` `fex(x):=''expkmax;`	
`ka[k]:=rat((taylor(((ll/q)^k*fex(ll/q)-(ll/p)^k*fex(ll/p))` `/((ll/q)*fex(ll/q)-(ll/p)*fex(ll/p)),ll,0,kmax)));`	K_k
`phil:1/(1+2*ppl[0]+2*ka[0]*ppl[1]);`	$\varphi(\lambda)$
`phiz:rat(expand(1-2*phil*(ppz[0]+ka[0]*ppz[1])));` `phi(z):=''phiz;`	$\varphi(z)$
`aal:taylor((1/q-1/p+phi(ll/q)-q/p*phi(2*ll))` `/(1/q*fex(-ll/q)-1/p*fex(ll/p)),ll,0,kmax+1);`	$A(\lambda)$
`ew:rat(expand(1+1/ll*q*ssl(rat(expand` `(z*phi(rat(ll)+p*rat(z)))))-aal/ll*ppl[1]));`	$E(W)$

The above sequence of commands is a MACSYMA program for computing the quantities $E(L)$, $E(W)$. These quantities are determined to order $O(\lambda^{kmax})$.

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